>> chapter 2. 本章习题.

6. 验证一个代数等式 ⇒ 直接验证. 左侧: v^μ

右侧: $v = v^\nu X_\nu = v^\nu \cdot \frac{\partial}{\partial x^\nu}$ 作用于 x^μ 上, ⇒ $v(x^\mu) = v^\nu \cdot \delta^\mu_\nu = v^\mu$ 得证.

⇒ 结论: 矢量在坐标基底下的第 μ 分量、等于将其作用在 x^μ 上的值.

8.(b). $[u,v](f) = u(v(f)) - v(u(f))$ * w 作用在 $[u,v](f)$ 上可以视作 $[u,v](f)$ 是个标量场. 从而 w 作用在标量场上.

$[[u,v],w](f) = [u(v(w(f))) - v(u(w(f)))] - w(u(v(f)) - v(u(f)))$

$= u(v(w(\cancel{f}))) - v(u(w(f))) - w(u(v(f))) + w(v(u(f)))$

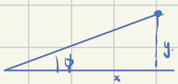
$[[w,u],v](f) = w(u(v(f))) - u(w(v(f))) - v(w(u(f))) + v(u(w(f)))$

$[[v,w],u](f) = v(w(u(f))) - w(v(u(f))) - u(v(\cancel{w}(f))) + u(w(v(f)))$

以上相加相消得

从而得到结论: Jacobi 恒等式. $[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$.

9.(a).



利用 7 的结论. 求 $\frac{\partial}{\partial x}$ 前的量只需作用在 x 上 ⇒.

$$\begin{cases} \frac{\partial x}{\partial r} = \frac{\partial(r\cos\phi)}{\partial r} = \cos\phi \\ \frac{\partial y}{\partial r} = \frac{\partial(r\sin\phi)}{\partial r} = \sin\phi \end{cases} \Rightarrow \underline{\frac{\partial}{\partial r} = \cos\phi\left(\frac{\partial}{\partial x}\right) + \sin\phi\left(\frac{\partial}{\partial y}\right)}.$$

$$\begin{cases} \frac{\partial}{\partial\phi}(x) = \frac{\partial(r\cos\phi)}{\partial\phi} = -r\cdot\sin\phi. \\ \frac{\partial}{\partial\phi}(y) = \frac{\partial(r\sin\phi)}{\partial\phi} = r\cdot\cos\phi. \end{cases} \Rightarrow \frac{\partial}{\partial\phi} = -r\cdot\sin\phi\left(\frac{\partial}{\partial x}\right) + r\cos\phi\left(\frac{\partial}{\partial y}\right).$$

* 注: 这本书习惯与正常不同

这是 $\frac{\partial f}{\partial x\partial y}$ 表示先对 x 偏, 再对 y 偏. 从下一题起修改.

(b).

$$\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}\right] = \frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial r}\right). \qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial r}.$$

$$= \frac{\partial x}{\partial r}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left[\frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}\right]$$

$$= \frac{\partial x}{\partial r}\cdot\frac{\partial^2}{\partial^2 x} + \frac{\partial y}{\partial r}\cdot\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\right) - \frac{\partial^2 x}{\partial r\partial x}\frac{\partial}{\partial x} - \frac{\partial x}{\partial r}\cdot\frac{\partial^2}{\partial x^2} - \frac{\partial y}{\partial r\partial x}\frac{\partial}{\partial y} - \frac{\partial y}{\partial r}\frac{\partial^2}{\partial y\partial x}$$

将它作用在 x 上. $x = r\cos\theta$, $y = r\cdot\sin\theta$

$\frac{\partial^2 x}{\partial^2 x} = 0$ ⇓ 第一项为0.

$\frac{\partial x}{\partial x} = 1$, $\frac{\partial 1}{\partial y} = 0$ ⇓ III 为 0.

$\frac{\partial^2 x}{\partial r\partial x} = \frac{\partial^2(r\cos\phi)}{\partial r\partial x} = \frac{\partial(\cos\phi)}{\partial x}$. 而 $\cos\phi = \frac{x}{\sqrt{x^2+y^2}}$

$\Rightarrow \frac{\partial\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^{3/2}}$ 而 $\frac{\partial x}{\partial x} = 1$. ⇒ III $= \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{r^2}\frac{1}{r} = \frac{1}{r}\cdot\sin^2\phi$.

IV $= \frac{\partial^2 x}{\partial x^2} = 0$ ⇒ IV $= 0$

VI $= \frac{\partial^2 x}{\partial x\partial y} = 0$ ⇒ VI $= 0$

V $= \frac{\partial x}{\partial y} = 0$ ⇒ V $= 0$.

$$\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}\right](x) = \frac{1}{r}\cdot\sin^2\phi.$$

再将它作用在y上. I. $\frac{\partial^2 y}{\partial x^2}=0$. II. $\frac{\partial y}{\partial x}=0$. III. $\frac{\partial y}{\partial x}=0$. IV. $\frac{\partial^2 y}{\partial x^2}=0$. V. $\frac{\partial y}{\partial y}=1$. VI. $\frac{\partial^2 y}{\partial y\partial x}=0$

$$\frac{\partial y}{\partial r\partial x} = \frac{\partial(r\cdot\sin\phi)}{\partial r\cdot\partial x} = \frac{\partial(\sin\phi)}{\partial x} = \frac{\partial\left(\frac{y}{\sqrt{x^2+y^2}}\right)}{\partial x} = -\frac{y}{(x^2+y^2)^{3/2}} = -\frac{y}{r^3}$$

从而展开的结果为 $\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}\right] = -\frac{y^2}{r^3}\cdot\left(\frac{\partial}{\partial x}\right) - \frac{y}{r^3}\left(\frac{\partial}{\partial y}\right).$

(c).

$$[\hat{e}_r, \hat{e}_\phi] = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \phi}\right]$$

$$= \left[\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}, -\sin\phi\frac{\partial}{\partial x} + \cos\phi\frac{\partial}{\partial y}\right]$$

$$= \left[\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}, -\sin\phi\frac{\partial}{\partial x}\right] + \left[\cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y}, +\cos\phi\frac{\partial}{\partial y}\right].$$

$$= \left[\cos\phi\frac{\partial}{\partial x}, -\sin\phi\frac{\partial}{\partial x}\right] + \left[\sin\phi\frac{\partial}{\partial y}, -\sin\phi\frac{\partial}{\partial x}\right] + \left[\cos\phi\frac{\partial}{\partial x}, \cos\phi\frac{\partial}{\partial y}\right] + \left[\sin\phi\frac{\partial}{\partial y}, \cos\phi\frac{\partial}{\partial y}\right].$$

$$= \cos\phi\frac{\partial}{\partial x}\left(-\sin\phi\cdot\frac{\partial}{\partial x}\right) - \left(-\sin\phi\frac{\partial}{\partial x}\left(\cos\phi\frac{\partial}{\partial x}\right)\right)$$
$$+ \sin\phi\frac{\partial}{\partial y}\left(-\sin\phi\frac{\partial}{\partial x}\right) - \left(-\sin\phi\frac{\partial}{\partial x}\left(\sin\phi\frac{\partial}{\partial y}\right)\right).$$
$$+ \cos\phi\frac{\partial}{\partial x}\left(\cos\phi\frac{\partial}{\partial y}\right) - \cos\phi\frac{\partial}{\partial y}\left(\cos\phi\cdot\frac{\partial}{\partial x}\right)$$
$$+ \sin\phi\cdot\frac{\partial}{\partial y}\left(\cos\phi\cdot\frac{\partial}{\partial y}\right) + \cos\phi\frac{\partial}{\partial y}\left(\sin\phi\frac{\partial}{\partial y}\right).$$

有些重复

$$\frac{\partial}{\partial x}(\sin\phi) = \frac{\partial}{\partial x}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{\left(-\frac{1}{2}\right)\cdot 2y}{(x^2+y^2)^{3/2}} = \frac{-y}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial}{\partial y}(\sin\phi) = \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{x^2+y^2}} + \frac{-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial}{\partial x}(\cos\phi) = \frac{y^2}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial}{\partial y}(\cos\phi) = -\frac{x}{(x^2+y^2)^{3/2}}$$

从而 上面作用在x上. 会有

$$-\cos\phi\left(-\frac{y}{(x^2+y^2)^{3/2}}\right) + \cdots\cdots$$

作用在y上, 会有……

这就是两个分量.

∴ $\left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \phi}\right] \neq 0$. 且 $\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial x}\right] \neq 0$.

10. 假设答案式如此. 则 $[u,v] = \left(u^{\gamma}\cdot\frac{\partial v^{\mu}}{\partial x^{\gamma}} - v^{\gamma}\cdot\frac{\partial u^{\mu}}{\partial x^{\gamma}}\right)\cdot\left(\frac{\partial}{\partial x^{\mu}}\right)$.

将 u, v 在坐标基底下展开. 并强行作用在 f 上.

$$-[u^{\mu}\cdot\frac{\partial}{\partial x^{\mu}},\ v^{\gamma}\cdot\frac{\partial}{\partial x^{\gamma}}](f)$$

$$= v^{\gamma}\frac{\partial}{\partial x^{\gamma}}\left(u^{\mu}\frac{\partial f}{\partial x^{\mu}}\right) - u^{\mu}\frac{\partial}{\partial x^{\mu}}\left(v^{\gamma}\cdot\frac{\partial f}{\partial x^{\gamma}}\right)$$

$$= v^{\gamma}\left(\frac{\partial u^{\mu}}{\partial x^{\gamma}}\cdot\frac{\partial f}{\partial x^{\mu}} + u^{\mu}\cdot\frac{\partial^2 f}{\partial x^{\gamma}\partial x^{\mu}}\right) - u^{\mu}\left(\frac{\partial v^{\gamma}}{\partial x^{\mu}}\cdot\frac{\partial f}{\partial x^{\gamma}} + v^{\gamma}\cdot\frac{\partial f}{\partial x^{\mu}\partial x^{\gamma}}\right)$$ 由于坐标基底是对易的 $\Rightarrow$

$$= v^{\gamma}\cdot\frac{\partial u^{\mu}}{\partial x^{\gamma}}\cdot\frac{\partial f}{\partial x^{\mu}} - u^{\mu}\frac{\partial v^{\gamma}}{\partial x^{\mu}}\cdot\frac{\partial f}{\partial x^{\gamma}}$$ 调整分量的角标 $\Rightarrow$.

$$= v^{\gamma}\cdot\frac{\partial u^{\mu}}{\partial x^{\gamma}}\cdot\frac{\partial f}{\partial x^{\mu}} - u^{\gamma}\cdot\frac{\partial v^{\mu}}{\partial x^{\gamma}}\cdot\frac{\partial f}{\partial x^{\mu}}$$

从而得证.

11. 物理意义: 对偶矢的第 ν 分量, 等于它作用在第 ν 个基矢上所得值.

< 矢量的第 ν 分量, 等于它作用在第 ν 个对偶基矢上所得值.

↳ 实际上是第 ν 个对偶基矢作用在 v 上. e.g. $dx^{\mu}(v) = v(x^{\mu})$. 这是上面小题 6) 的结论.

张量分量的计算方法由其"进化"而来.

证对偶矢量等式, 只需将其作用在基矢上. $\omega = \omega(e_{\mu})\, e^{\mu *}$

$\omega(e_{\gamma}) = \omega(e_{\mu})\cdot e^{\mu *}(e_{\gamma}) = \omega(e_{\mu})\cdot\delta^{\gamma}{}_{\mu} = \omega(e_{\gamma})$.

设 $v = v^{\gamma}\cdot e_{\gamma}$. $e^{\mu *}(v) = e^{\mu *}(v^{\gamma}\cdot e_{\gamma}) = v^{\gamma}\cdot\delta^{\gamma}{}_{\gamma} = v_{\mu}$ $\Rightarrow v_{\mu} = e^{\mu *}$. 得证.

12. 对偶矢的变换关系. 考虑任一对偶矢 df. 它的分量: $df = \frac{\partial f}{\partial x^{\mu}}\cdot dx^{\mu} = \frac{\partial f}{\partial x'^{\mu}}\cdot dx'^{\mu} = \frac{\partial f}{\partial x^{\mu}}\cdot\frac{\partial x^{\mu}}{\partial x'^{\gamma}}\cdot dx'^{\gamma}$

有了 $dx^{\mu} \Leftrightarrow dx'^{\mu}$ 变换关系, 问题解决.

14. 设矢量基底间存在变换. $e'_{\mu} = A^{\gamma}{}_{\mu}e_{\gamma}$. 则对偶基底间: $e'^{\mu *} = (\tilde{A}^{-1})_{\gamma}{}^{\mu}\, e^{\gamma *}$.

把需要的写下来. $T(e'^{\mu *}, \cdot, e'_{\mu}) = T((\tilde{A}^{-1})_{\gamma}{}^{\mu}e^{\gamma *}, \cdot, A^{\sigma}{}_{\mu}\cdot e_{\sigma})$.

$$= (\tilde{A}^{-1})_{\gamma}{}^{\mu}A^{\sigma}{}_{\mu}T(e^{\gamma *}, \cdot, e_{\sigma})$$

$$= (A^{-1})^{\mu}{}_{\gamma}\cdot A^{\sigma}{}_{\mu}\ T(e^{\gamma *}, \cdot, e_{\sigma}).$$

$$= \delta_{\gamma}{}^{\sigma}\ T(e^{\gamma *}, \cdot, e_{\sigma}) = T(e^{\gamma *}, \cdot, e_{\gamma}).$$ 得证.

16. 设曲线参数方程为 $X^\mu(t)$. 重参化后为 $X'^\mu(t')$

重参化前的线长: $l=\int\sqrt{g(T,T)}\,dt$

$$=\int\sqrt{g\left[\frac{dx^\mu(t)}{dt}\left(\frac{\partial}{\partial x^\mu}\right),\ \frac{dx^\nu(t)}{dt}\left(\frac{\partial}{\partial x^\nu}\right)\right]}\,dt$$

$$=\int\sqrt{\frac{dx^\mu(t)}{dt}\cdot\frac{dx^\nu(t)}{dt}\,g\left[\left(\frac{\partial}{\partial x^\mu}\right),\left(\frac{\partial}{\partial x^\nu}\right)\right]}\,dt$$

$$=\int\sqrt{dx^\mu(t)\cdot dx^\nu(t)\cdot g[(\partial),(\partial)]}$$

重参化后线长: $l=\int\sqrt{g(T',T')}\,dt'$

$$=\int\sqrt{g\left[\frac{dx'^\mu(t)}{dt'}e_\mu,\ \frac{dx'^\nu(t)}{dt'}e_\nu\right]}\,dt'$$

$$=\int\sqrt{g\left[\frac{dx'^\mu(t)}{dt}\cdot\frac{dt}{dt'}e_\mu,\ \frac{dx'^\nu(t)}{dt}\cdot\frac{dt}{dt'}e_\nu\right]}\,dt'$$

$$=\int\sqrt{g\left[dx'^\mu(t)\,e_\mu,\ dx'^\nu(t)\,e_\nu\right]}$$

$$=\int\sqrt{dx^\mu(t)\cdot dx^\nu(t)\,g[(),()]}$$

从而得证.

19. a) 球坐标系与平行坐标系间的关系取为:

$$\begin{cases}x=r\cdot\sin\theta\cos\phi\\ y=r\cdot\sin\theta\cdot\sin\phi\\ z=r\cdot\cos\theta\end{cases}$$

写下Jacobi阵

$$\begin{bmatrix}\sin\theta\cos\phi & r\cdot\cos\theta\cdot\cos\phi & -r\cdot\sin\theta\cdot\sin\phi\\ \sin\theta\sin\phi & r\cdot\cos\theta\cdot\sin\phi & r\cdot\sin\theta\cdot\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{bmatrix}$$

由张量变换律 $g'_{\mu\nu}=\frac{\partial x^\rho}{\partial x'^\mu}\cdot\frac{\partial x^\sigma}{\partial x'^\nu}\cdot g_{\rho\sigma}$ 而欧氏度规 $g_{\rho\sigma}$ 是个对角度规

$$g'_{rr}=\frac{\partial x}{\partial r}\frac{\partial x}{\partial r}+\frac{\partial y}{\partial r}\frac{\partial y}{\partial r}+\frac{\partial z}{\partial r}\frac{\partial z}{\partial r}$$

$$=\sin^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi+\cos^2\theta=1$$

$$g'_{\theta\theta}=\frac{\partial x}{\partial\theta}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial y}{\partial\theta}\cdot\frac{\partial y}{\partial\theta}+\frac{\partial z}{\partial\theta}\cdot\frac{\partial z}{\partial\theta}$$

$$=r^2\cdot\cos^2\theta\cos^2\phi+r^2\cdot\cos^2\theta\sin^2\phi+r^2\sin^2\theta=r^2.$$

$$g'_{\phi\phi}=\frac{\partial x}{\partial\phi}\cdot\frac{\partial x}{\partial\phi}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\phi}+\frac{\partial z}{\partial\phi}\cdot\frac{\partial z}{\partial\phi}.$$

$$=r^2\cdot\sin^2\theta\sin^2\phi+r^2\cdot\sin^2\theta\cdot\cos^2\phi=r^2\cdot\sin^2\theta.$$

$$g'_{r\theta}=\frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial\theta}+\cdots+\cdots$$

$$=r\cdot\sin\theta\cdot\cos\theta\cdot\cos^2\phi+r\cdot\sin\theta\cos\theta\sin^2\phi-r\cdot\sin\theta\cos\theta=0.$$

$$g'_{r\phi}=\frac{\partial x}{\partial r}\cdot\frac{\partial x}{\partial\phi}+\cdots+\cdots$$

$$=-r\sin^2\theta\sin\phi\cos\phi+r^2\cdot\sin^2\theta\sin\phi\cos\phi+0=0$$

$$g'_{\theta\phi}=\frac{\partial x}{\partial\theta}\cdot\frac{\partial x}{\partial\phi}+\cdots+\cdots$$

$$=-r\sin\theta\sin\phi\cos\theta\cos\phi+r\cdot\sin\theta\sin\phi\cos\theta\cos\phi=0.$$

b). 不会了.

20. 由 19(a). $\|\frac{\partial}{\partial r}\| = \sqrt{g(\frac{\partial}{\partial r}, \frac{\partial}{\partial r})} = \sqrt{g_{rr}} = 1$，其余同理.

21. 要求张量的分量，只需将张量作用在相应基矢、对偶基矢上.

$$T'^{\mu}{}_{\nu} = T^a{}_b \, (dx'^{\mu})_a \left(\frac{\partial}{\partial x'^{\nu}}\right)^b$$ ⇒ 利用矢量与对偶矢量的坐标基底变换.

$$= T^a{}_b \left(\frac{\partial x'^{\mu}}{\partial x^{\rho}}\right)(dx^{\rho})_a \left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right)\left(\frac{\partial}{\partial x^{\sigma}}\right)^b$$

$$= \left(\frac{\partial x'^{\mu}}{\partial x^{\rho}}\right)\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\right) T^a{}_b \, (dx^{\rho})_a \left(\frac{\partial}{\partial x^{\sigma}}\right)^b = (\ )(\ ) \cdot T^{\rho}{}_{\sigma}.$$

23.

1). $\left(\frac{\partial}{\partial x^{\mu}}\right)^a g_{ab} \left(\frac{\partial}{\partial x^{\nu}}\right)^b = g_{ab} \left(\frac{\partial}{\partial x^{\mu}}\right)^a \left(\frac{\partial}{\partial x^{\nu}}\right)^b = g_{\mu\nu}$. 作用！ 故正确

2). $g^{ab} \, (dx^{\mu})_b \, (dx^{\nu})_a = g^{\mu\nu}$. 故正确

3). $g_{ab} \left(\frac{\partial}{\partial x^{\mu}}\right)^b$ 不妨将其作用到矢量 $\left(\frac{\partial}{\partial x^{\nu}}\right)^a$ 上.

⇒ $g_{ab} \left(\frac{\partial}{\partial x^{\mu}}\right)^b \left(\frac{\partial}{\partial x^{\nu}}\right)^a = g_{\nu\mu}$. 而 $(dx^{\mu})_a \left(\frac{\partial}{\partial x^{\nu}}\right)^a = \delta^{\mu}{}_{\nu}$. 故错误

4). $g^{ab} \, (dx^{\mu})_b$. 不妨将其再作用到一个对偶矢 $(dx^{\nu})_a$ 上.

⇒ $g^{ab} \, (dx^{\mu})_b \, (dx^{\nu})_a = g^{\nu\mu}$.

而 $\left(\frac{\partial}{\partial x^{\mu}}\right)^a (dx^{\nu})_a = \delta^{\nu}{}_{\mu}$. 故错误.

5). v^{μ} = 矢量分量. v_{μ} = 用度规降指标后所得对偶矢的分量.

设 $v^a = v^{\mu} (e_{\mu})^a$. 降指标: $v_b = v^{\mu} g_{ab} (e_{\mu})^a$ 求对偶矢分量 ⇒ 将之作用在矢量上. $v_b \, ((e_{\sigma})^b) = v^{\mu} g_{ab} (e_{\mu})^a (e_{\sigma})^b = v^{\mu} g_{\mu\sigma}$ ⇒ $v_{\sigma} = v^{\mu} g_{\mu\sigma}$

设 $w_a = w_{\mu} (e^{\mu})_a$. 升指标: $w^b = w_{\mu} \, g^{ab} (e^{\mu})_a$. $w^b \, ((e^{\sigma})_b) = w_{\mu} \, g^{ab} (e^{\mu})_a (e^{\sigma})_b = w_{\mu} \, g^{\mu\sigma}$ ⇒ $v_{\sigma} \, w^{\sigma} = v^{\mu} \, w_{\mu}$. 正确.

6). $g_{\mu\nu} \, T^{\nu\rho} \, S_{\rho}{}^{\sigma}$
$$= g_{\mu\nu} \, g^{\nu\alpha} \, T_{\alpha}{}^{\rho} \, S_{\rho}{}^{\sigma}$$
$$= g_{\mu\nu} \, g^{\nu\alpha} \, g^{\rho\beta} \, T_{\alpha\beta} \, S_{\rho}{}^{\sigma}$$
$$= \underbrace{g_{\mu\nu} \, g^{\nu\alpha}}_{\delta_{\mu}{}^{\alpha}} \, \underbrace{g^{\rho\beta}}_{g^{\beta\rho}} \, T_{\alpha\beta} \, g_{\rho\omega} \, S^{\omega\sigma}$$

(度规对称).

$\delta^{\beta}{}_{\omega}$

故正确.

(7). 作用以对偶矢验证. 左: $v^a u^b \, (dx^{\mu})_a (dx^{\nu})_b = v^{\mu} u^{\nu}$

右: $v^b u^a \, (dx^{\mu})_a (dx^{\nu})_b = v^{\nu} u^{\mu}$ 故不正确.

角标指定了作用顺序.

(8). 与(7)同理. 正确.