Day 7.

虚位移原理.

将迫使力学系遵从约束的力称为约束力 (Constrained Force). 而其余的力则称为主动力 (Applied Force). 将主动力记为 $\vec{F}_{(\alpha)}$，约束力记为 $\vec{N}_{(\alpha)}$. 定义理想约束为所有质点所受约束力虚功之和为0的约束：$\sum_\alpha \vec{N}_{(\alpha)}\cdot\delta\vec{x}_{(\alpha)}=0$

从牛顿力学出发，我们有 $\vec{F}_{(\alpha)}+\vec{N}_{(\alpha)}=m_{(\alpha)}\ddot{\vec{x}}_{(\alpha)}$. 从而有 $\sum_{\alpha=1}^{N}(\vec{F}_{(\alpha)}+\vec{N}_{(\alpha)}-m_{(\alpha)}\ddot{\vec{x}}_{(\alpha)})\cdot\delta\vec{x}_{(\alpha)}=0$. $\Rightarrow \sum_{\alpha=1}^{N}(\vec{F}_{(\alpha)}-m_{(\alpha)}\ddot{\vec{x}}_{(\alpha)})\cdot\delta\vec{x}_{(\alpha)}=0$ 故平衡态有 $\sum_\alpha \vec{F}_{(\alpha)}\cdot\delta\vec{x}_{(\alpha)}=0$.

(几何：理想约束的约束力正交于构形流形).

下面从 d'Alembert's Principle 推出拉氏方程. 由于各粒子的虚位移并不独立，则不可直接得 $\vec{F}_{(\alpha)}=m_{(\alpha)}\ddot{\vec{x}}_{(\alpha)}$. 故应先用广义坐标将上式拆成 s 个独立方程: $\delta\vec{x}_{(\alpha)}=\frac{\partial x_{(\alpha)}}{\partial q^a}\delta q^a$.

$\Rightarrow \sum_\alpha(F_{(\alpha)}-m_{(\alpha)}\ddot{x}_{(\alpha)})\frac{\partial x_{(\alpha)}}{\partial q^a}\cdot\delta q^a=0$. $\Rightarrow \sum_\alpha m_{(\alpha)}\ddot{x}_{(\alpha)}\cdot\frac{\partial x_{(\alpha)}}{q^a}=\sum_\alpha F_{(\alpha)}\cdot\frac{\partial x_{(\alpha)}}{\partial q^a}$ $a=1,\dots,s$. 设法将这 s 个方程中所有项全用广义坐标表示.

$\dot{x}_{(\alpha)}=\frac{\partial x_{(\alpha)}}{\partial q^a}\cdot\dot{q}^a+\frac{\partial x_{(\alpha)}}{\partial t}$. $\Rightarrow \frac{\partial\dot{x}_{(\alpha)}}{\partial\dot{q}^a}=\frac{\partial x_{(\alpha)}}{q^a}$. 将 s 个方程变形:

左边 $=\frac{d}{dt}\left(\sum_\alpha m_{(\alpha)}\dot{x}_{(\alpha)}\cdot\frac{\partial x_{(\alpha)}}{\partial q^a}\right)-\sum_\alpha m_{(\alpha)}\cdot\dot{x}_{(\alpha)}\cdot\frac{d}{dt}\cdot\frac{\partial x_{(\alpha)}}{\partial q^a}$

(利用交换求导次序). $=\frac{d}{dt}\left(\sum_\alpha m_{(\alpha)}\dot{x}_{(\alpha)}\cdot\frac{\partial\dot{x}_{(\alpha)}}{\partial\dot{q}^a}\right)-\sum_\alpha m_{(\alpha)}\cdot\dot{x}_{(\alpha)}\cdot\frac{\partial\dot{x}_{(\alpha)}}{\partial q^a}$

(凑微分) $=\frac{d}{dt}\cdot\frac{\partial}{\partial\dot{q}^a}\left(\frac{1}{2}\sum_\alpha m_{(\alpha)}\cdot\dot{x}_{(\alpha)}^2\right)-\frac{\partial}{\partial q^a}\left(\frac{1}{2}\cdot\sum_\alpha m_{(\alpha)}\cdot\dot{x}_{(\alpha)}^2\right)$

$=\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}^a}\right)-\frac{\partial T}{\partial q^a}$.

而将右侧定义为广义力 $Q_a\equiv\sum_\alpha F_{(\alpha)}\cdot\frac{\partial x_{(\alpha)}}{\partial q^a}$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}^a}\right)-\frac{\partial T}{\partial q^a}=Q_a$. 对于所有力均为保守力的情形，$F_\alpha=-\frac{\partial V}{\partial x_{(\alpha)}}$.

从而 $Q_a=-\sum_\alpha\frac{\partial V}{\partial x_{(\alpha)}}\frac{\partial x_{(\alpha)}}{\partial q^a}=-\frac{\partial V}{\partial q^a}$. 立刻得到标准的拉格朗日方程.

→ Jourdain's Principle: 虚功率: $\sum_\alpha[F_{(\alpha)}-m_{(\alpha)}\ddot{x}_{(\alpha)}]\cdot\delta x_{(\alpha)}=0$.

→ 便于计算机求解的高斯最小约束：$Z=\sum_\alpha\frac{1}{2m_{(\alpha)}}(\vec{F}_{(\alpha)}-m_{(\alpha)}\ddot{x}_{(\alpha)})^2$ 最小.