Day 1 变分法

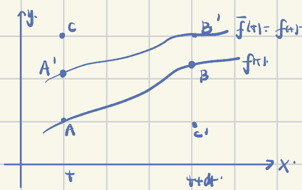
函数：在两个集合间建立的映射。 $f: t \to y = f(t)$.

泛函：函数集合到复数集C上的映射。 $f \mapsto S = S[f]$　通常而言，泛函是通过积分构造的：$S[f] = \int_{t_1}^{t_2} dt\, L(t, f(t), f'(t), \cdots)$

"输入的无穷小变化引起输出无穷小变化，对函数而言是"微分"，而对泛函则是"变分""（注意：函数自身发生了无穷小变化）。 $\Rightarrow \delta f(t) = \tilde{f}(t) - f(t)$.

变分的运算规则：1). $\delta(af_1 + bf_2) = a\cdot\delta f_1 + b\cdot\delta f_2$.　2). $\delta(f_1 f_2) = (\delta f_1) f_2 + f_1\,\delta f_2$.　3). 变分和微分可以交换. $\delta(df) = d(\delta f)$

下面来证明3).



如图，要求 $\tilde{f}(t+dt) - f(t)$，有两种路径来计算它的值。

1). A→B→B'.

C'B的长度：$f(t+dt) - f(t) = df(t)$.

BB'的长度：$\tilde{f}(t+dt) - f(t+dt) = \delta(f(t+dt)) = \delta(f(t) + df(t)) = \delta f(t) + \delta(df(t))$.

2). A→A'→B'

AA'的长度：$\tilde{f}(t) - f(t) = \delta f(t)$.

A'C的长度：$\tilde{f}(t+dt) - \tilde{f}(t) = d(\tilde{f}(t)) = d[(f+\delta f)(t)] = df(t) + d(\delta f(t))$　⇒ 得证.

从而达到得到：变分和求导可以交换顺序：$\delta(\frac{d}{dt} f(t)) = \frac{d}{dt}(\delta f(t))$.

下面我们讨论泛函的导数。对于一般的函数，我们有：$f(t) = f(t + \varepsilon\cdot dt) = f(t) + (\varepsilon t)\cdot\frac{df(t)}{dt} + \frac{1}{2}(\varepsilon t)^2\cdot\frac{d^2f(t)}{dt^2}\cdots$　$df(t)$，$d^2f(t)$ 称为 $f(t)$ 的1,2,…阶微分。

对于泛函. $S[\tilde{f}] = S[f + \delta f] = S[f] + \varepsilon\cdot\delta S[f] + \frac{1}{2}\varepsilon^2\,\delta^2 S[f] + \frac{1}{6}\varepsilon^2\,\delta^3 S[f] + \cdots$　$\delta^n S[f]$ 被称为泛函的n阶变分.

泛函变分是什么？不妨把它和多元函数的微分作类比，从而给出定义：$dF = \sum_n \frac{\partial F}{\partial x_n}\cdot dx_n$

在t处改变 $\delta f(t)$，对S的影响.

$\delta S = \int dt\cdot\frac{\delta S}{\delta f(t)}\cdot\delta f(t)$. 自变量.　我们可以写出更高阶的泛函导：$\delta^2 S[f] = \int dt_1 \int dt_2\,\frac{\delta^2 S}{\delta f(t_1)\,\delta f(t_2)}\cdot\delta f(t_1)\cdot\delta f(t_2)$.

一阶泛函导数.　它的作用是将函数变分 $\delta f(t)$（无穷小的函数）→ δS（无穷小的数）.

但是如何给出泛函的导数？　注意到 $S[f + \varepsilon\cdot\delta f]$ 可以视作自变量为 ε 的函数.

$\Rightarrow$ Taylor Exp.　$S[f + \varepsilon\cdot\delta f] = S[f] + \varepsilon\cdot\frac{d}{d\varepsilon} S[f + \varepsilon\cdot\delta f]\Big|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2\,\frac{d^2}{d\varepsilon^2} S[f + \varepsilon\cdot\delta f]\Big|_{\varepsilon=0} + \cdots$

> 这里我们将变分和求导有类似的展开形式.
> 因此我们将变分用求导来表述.

对比一下 $S[f + \varepsilon\cdot\delta f]$ 的展开式，我们有：$\delta S = \frac{d}{d\varepsilon} S[f + \varepsilon\cdot\delta f]\Big|_{\varepsilon=0} = \int dt\,\frac{\delta S}{\delta f(t)}\cdot\delta f(t)$.

若泛函 $S = \int_{t_1}^{t_2} dt\cdot L(t, f, f', f'', \cdots)$.　$\to S[f + \delta f] = \int_{t_1}^{t_2} dt\cdot L(t, f+\delta f, f'+\delta f', f''+\delta f'', \cdots)$

→ 将各个 t 处 $L(t)$ 改变的贡献加起来。

从而我们有: $\delta S = \int_{t_1}^{t_2} dt \cdot \Big(\underbrace{\frac{\partial L}{\partial f} \cdot \delta f + \frac{\partial L}{\partial f'} \cdot \delta f' + \cdots}_{\delta L(t) = \tilde{L}(t) - L(t)} \Big)$. $\Rightarrow \delta S = \delta\left(\int_{t_1}^{t_2} dt \cdot L\right) = \int_{t_1}^{t_2} dt \cdot \delta L$.

然而我们应去掉 $\delta f'$, $\delta f''$ 等项, 这需使用分部积分法。

我们希望将这个式子写成 $\delta S = \int dt\,[\quad]\cdot\delta f$ 的形式, 那么我们应使用分部积分。

$$\frac{\partial L}{\partial f'}\cdot\delta f' = \frac{\partial L}{\partial f'}\,\frac{d}{dt}(\delta f) = \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\cdot\delta f\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right)$$

(变分、求导交换)

对于更高阶的情形, 只需分部积分更多次。

$$\begin{aligned}\frac{\partial L}{\partial f''}\cdot\delta f'' = \frac{\partial L}{\partial f''}\cdot\frac{d^2}{dt^2}\delta f &= \frac{d}{dt}\left(\frac{\partial L}{\partial f''}\cdot\frac{d}{dt}\delta f\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial f''}\right)\cdot\frac{d}{dt}(\delta f) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial f''}\cdot\frac{d}{dt}\delta f\right) - \frac{d}{dt}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial f''}\right)\cdot(\delta f)\right] + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial f''}\right)\cdot\delta f \\ &= \frac{d}{dt}\left[\frac{\partial L}{\partial f''}\cdot\frac{d}{dt}\delta f - \frac{d}{dt}\left(\frac{\partial L}{\partial f''}\right)\cdot\delta f\right] + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial f''}\right)\delta f\end{aligned}$$

这样做的"代价"是出现了一个全导项。于是我们将泛函的变分写成: (对于这种特殊形式的泛函)。

$$\delta S = \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial f} - \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial f''}\right) + \cdots\right]\delta f + B\Big|_{t_1}^{t_2}$$ "boundary term"

注意到在 L 包含 f 的 n 阶导时 $\Leftrightarrow$ 在边界项中包含 $\delta f(t)$ 的 $n-1$ 阶导数。因此我们在变分法中假设 $B\big|_{t_1}^{t_2}=0 \Leftarrow \delta f\big|_{t_1} = \delta f\big|_{t_2} = 0 \quad \cdots \quad \delta f^{(n-1)}\big|_{t_1} = \delta f^{(n-2)}\big|_{t_2} = 0$.

从而若我们在 L 上加上一个函数 $F = F(t, f, f', \dots f^{(n-1)})$ 对时间全导数, 并不会影响泛函导数的计算结果。

下面考虑如何使泛函取到极值。若 $S[f]$ 在 $f = f(t)$ 时取极值, 这等价于对 $\forall\delta f$, 函数 $S[f+\varepsilon\delta f]$ 在 $\varepsilon=0$ 时取极值。

利用泛函导数的定义立刻有: $\delta S[f] = \int dt \cdot \frac{\delta S[f]}{\delta f}\Big|_f \delta f(t) = \frac{dS[f+\varepsilon\delta f]}{d\varepsilon}\Big|_{\varepsilon=0} = 0$. $\Rightarrow \delta S[f]=0$, 或 $\frac{\delta S[f]}{\delta f}=0$. $\Rightarrow$ 在任何位置的扰动 δf, S 都不变。

对于最常见的一类泛函, 我们有: $S[f] = \int dt\, L(t, f(t), f'(t))$. 从而立刻有: $-\frac{\delta S}{\delta f} = \frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right) - \frac{\partial L}{\partial f} = 0$. 此即所谓 E-L 方程。

考虑对 L 求全导: $\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial f}\cdot f' + \frac{\partial L}{\partial f'}\cdot f'' = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial f} f' + \frac{d}{dt}\left(\frac{\partial L}{\partial f'} f'\right) = \frac{\partial L}{\partial t} - \left(\frac{d}{dt}\left(\frac{\partial L}{\partial f'}\right) - \frac{\partial L}{\partial f}\right) f' + \frac{d}{dt}\left(\frac{\partial L}{\partial f'} f'\right)$.

从而有: $\frac{d}{dt}\left(\frac{\partial L}{\partial f'} f' - L\right) + \frac{\partial L}{\partial t} = 0$. $\Rightarrow$ 在 $\frac{\partial L}{\partial t}=0$ (L 不显含 t) 时, 有 $\frac{\partial L}{\partial f'} f' - L = 0$.

对于更一般的情形, 极值条件为: $\frac{\partial S}{\partial f} = \sum_{n=0} (-1)^n \frac{d^n}{dt^n}\left(\frac{\partial L}{\partial f^{(n)}}\right) = 0$

对于多个函数作为泛函自变量的情形: $\delta S = \int dx \left(\frac{\partial S}{\partial f_1}\cdot\delta f_1 + \frac{\partial S}{\partial f_2}\cdot\delta f_2 + \cdots\right) = 0$.

对于多元函数的情形, 我们看一个最简单例: 令 $f = f(t,x)$. $S[f] = \iint dt\cdot dx\; L\left(t, x, f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}\right)$.

$\Rightarrow \delta S = \iint dt\cdot dx\; \delta L\left(t, x, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial t}\right)$

$$= \iint dt\,dx \left[\frac{\partial L}{\partial f} \cdot \delta f + \frac{\partial L}{\partial \left(\frac{\partial f}{\partial x}\right)} \delta\left(\frac{\partial f}{\partial x}\right) + \frac{\partial L}{\partial \left(\frac{\partial f}{\partial t}\right)} \cdot \delta\left(\frac{\partial f}{\partial t}\right) \right]$$

$$= \iint dt\,dx \left[\frac{\partial L}{\partial f} \cdot \delta f + \frac{\partial L}{\partial f_x} \cdot \frac{\partial \delta f}{\partial x} + \frac{\partial L}{\partial f_t} \cdot \frac{\partial \delta f}{\partial t} \right]$$

$$= \iint dt\,dx \left[\frac{\partial L}{\partial f} \delta f + \frac{\partial}{\partial x} \left[\frac{\partial L}{\partial f_x} \cdot \delta f \right] - \partial\left(\frac{\partial L}{\partial f_x}\right) \cdot \delta f + \cdots \right]$$

从而有PDE: $\frac{\partial S}{\partial f} = \frac{\partial L}{\partial f} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \left(\frac{\partial f}{\partial t}\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \left(\frac{\partial f}{\partial x}\right)}\right) = 0$ 成立.